

# The Master Field for the Half-Planar Approximation for Large $N$ Matrix Models and Boltzmann Field Theory

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In this talk results of study in various dimensions of the Boltzmann master field for a subclass of planar diagrams, so called half-planar diagrams, found in the recent work by Accardi, Volovich and one of us (I.A.) are presented.

## 1. Introduction

The problem of analytic summation of all planar diagrams in realistic models is still unsolved. Its solution is closely related with problem of finding the leading asymptotics in  $N \times N$  matrix models for large  $N$  and may have important applications to the hadron dynamics [1–4]. In the early 80-s it was suggested [3] that there exists a master field which dominates in the large  $N$  limit of invariant correlation functions of a matrix field.

The problem of construction of the master field has been discussed in many works, see for example [6]–[10]. Gopakumar and Gross [14] and Douglas [15] have constructed the master field for an arbitrary matrix model in terms of correlation functions. There has been a problem of construction an operator realization for the master field without knowledge of correlation functions. Recently this problem has been solved in [24] and it was shown that the master fields satisfy to standard equations of relativistic field theory but fields are quantized according to a new rule.

In this talk we are going to demonstrate that an operator realization for the master field for a subset of planar diagrams, so called half-planar (HP) diagrams, proposed in [22] gives an analytical summations of HP diagrams (see [25] for more details). This construction deals with the master field in a modified interaction representation in the free (Boltzmannian) Fock space. This new interaction representation involves not the ordinary exponential function of the interaction

but a rational function of the interaction. Corresponding correlation functions satisfy the Boltzmannian Schwinger-Dyson equations which are simpler than the usual Schwinger-Dyson equations. In particular in the case of quartic interaction one has a closed set of equations for two- and four-point correlation functions. We solve explicitly this system of equations. In the case of  $D$ -dimensional space-time we get a Bethe-Salpiter-like equation for the four-point correlation function. A special approximation reduces this system of integral equations to a linear integral equation which was considered [26] in the rainbow approximation in the usual field theory.

A solution of the Boltzmannian Schwinger-Dyson equations can be considered as a first non-trivial approximation to the planar correlation functions. Note in this context that in all previous attempts of approximated treatment of planar theory were used some non-perturbative approximation [11,12,27]. Topologically diagrams representing the perturbative series of Boltzmann correlation functions look as half-planar diagrams of the usual diagram technique for matrix models [22]. We compare numerically the two- and four-point HP correlation functions with the corresponding planar correlation functions for the one matrix model. For large variety of coupling constant the HP approximation reproduces the planar approximation with good accuracy. This fact gives us an optimism and makes sensitive a study of the HP approximation for realistic models.

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## 2. Half-Planar Approximation for the One Matrix Model

The master field in zero dimensional case is defined as  $\phi = a + a^+$ , where  $a$  and  $a^+$  satisfy the following relation

$$aa^+ = 1. \quad (2.1)$$

This algebra has a realization in the free (or Boltzmannian) Fock space [28] generated by the vacuum  $|0\rangle$ ,  $a|0\rangle = 0$ , and  $n$ -particle states  $|n\rangle = (a^+)^n|0\rangle$ .

A free  $n$ -point Green's function is defined as the vacuum expectation of  $n$ -th power of master field

$$G_n^{(0)} = \langle 0 | \phi^n | 0 \rangle. \quad (2.2)$$

As it is well-known, the Green's function (2.2) is given by a  $n$ -th moment of Wigner's distribution [5,16]

$$G_{2n}^{(0)} = \int_{-2}^2 \frac{d\lambda}{2\pi} \lambda^{2n} \sqrt{4 - \lambda^2} = \frac{(2n)!}{n!(n+1)!}.$$

This representation can be also obtained as a solution of the Schwinger-Dyson equations

$$G_{2n}^{(0)} = \sum_{m=1}^n G_{2m-2}^{(0)} G_{2n-2m}^{(0)}.$$

Interacting Green's functions are defined by the formula [22]

$$G_n = \langle 0 | \phi^n (1 + S_{int}(\phi))^{-1} | 0 \rangle. \quad (2.3)$$

In contrast to the ordinary quantum field theory where one deals with the exponential function of an interaction, here we deal with the rational function of an interaction. In [25] it was shown that under natural assumptions the form (2.3) is unique one which admits Schwinger-Dyson-like equations.

For the case of quartic interaction  $S_{int} = g\phi^4$  the Boltzmannian Schwinger-Dyson equations have the form

$$G_n = \sum_{l=1}^{k-1} G_{k-l-1}^{(0)} G_{l+n-k-1} + \quad (2.4)$$

$$\sum_{l=k+1}^n G_{l-k-1}^{(0)} G_{n+k-l-1} - g[G_{n-k}G_{k+2} +$$

$$G_{n-k+1}G_{k+1} + G_{n-k+2}G_k +$$

$$G_{n-k+3}G_{k-1}].$$

The distinguish feature of equations (2.4) is that for  $n \geq 4$  and  $2 \leq k \leq n-1$  the right hand side of (2.4) does not contain the Green's functions  $G_m$  with  $m > n$ . This fact permit us to write down a closed set of equations for any  $G_2$  and  $G_4$

$$G_2 = 1 - gG_2G_2 - gG_4,$$

$$G_4 = 2G_2 - 2gG_2G_4. \quad (2.5)$$

These equations follow from (2.4) for  $n = 2, k = 1$  and  $n = 4, k = 2$ . We set  $G_0 = 1$  to exclude vacuum insertions. From (2.5) one gets

$$2g^2G_2^3 + 3gG_2^2 + G_2 - 1 = 0. \quad (2.6)$$

Using the Cordano formula for  $g \leq \frac{\sqrt{3}}{18}$  we have

$$G_2 = \frac{1}{\sqrt{3}g} \text{Re} e^{i\frac{\pi}{6}} (\sqrt{1 - 108g^2} - i6\sqrt{3}g)^{\frac{1}{3}} -$$

$$\frac{1}{2g} = 1 - 3g + 16g^2 - 105g^3 + 768g^4 - \dots$$

Green's functions for  $n > 4$  are also determined from (2.4).

Now let us compare the Boltzmann theory and the planar approximation for the one-matrix model. Green's functions for the one-matrix model in the planar approximation are defined as

$$\Pi_{2n}(g) = \lim_{N \rightarrow \infty} \frac{1}{N^{1+n}} \frac{1}{\mathcal{Z}} \int DM \text{tr} (M^{2n})$$

$$\exp[-\frac{1}{2}\text{tr}(M^2) - \frac{g}{4N} \text{tr}(M^4)], \quad (2.7)$$

where  $\mathcal{Z}$  is a normalization factor. The integration in (2.7) is over  $N \times N$  hermitian matrices. According the 't Hooft diagram technique the perturbative expansion in the coupling constant of the correlation functions (2.7) is represented by a sum of all planar double-line graphs [1]. Due to the normalization factor  $N^{-(1+n)}$  external lines corresponding to  $\text{tr} M^{2n}$  can be treated as lines

$g$	$10^{-3}$	$10^{-2}$	$10^{-1}$	1	10	$10^2$	$10^3$
$\Pi_2$	$9.98 \cdot 10^{-1}$	$9.81 \cdot 10^{-1}$	$8.6 \cdot 10^{-1}$	$5.2 \cdot 10^{-1}$	$2.1 \cdot 10^{-1}$	$7.4 \cdot 10^{-2}$	$2.4 \cdot 10^{-2}$
$G_2$	$9.97 \cdot 10^{-1}$	$9.71 \cdot 10^{-1}$	$8.0 \cdot 10^{-1}$	$4.0 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$	$3.2 \cdot 10^{-2}$	$7.4 \cdot 10^{-3}$
$\Pi_4$	1.99	1.96	1.42	$4.84 \cdot 10^{-1}$	$7.87 \cdot 10^{-2}$	$9.26 \cdot 10^{-3}$	$9.76 \cdot 10^{-4}$
$G_4$	1.99	1.91	1.38	$4.43 \cdot 10^{-1}$	$7.16 \cdot 10^{-2}$	$8.65 \cdot 10^{-3}$	$9.37 \cdot 10^{-4}$

of a generalized vertex. We shall call two double-line planar graphs topologically equivalent if one of them can be transformed into the other by a continuous deformation on the plane.

A planar non-vacuum graph is an HP graph if it is topologically equivalent to a graph which can be drawn so that all its vertices lie on some plane line in the right of the generalized vertex  $\text{tr } M^{2n}$  and all propagators lie in the upper-half plane without overlapping [22]. Also we shall call a planar graph HP-irreducible if it is represented as an HP graph in a unique way. A simple analysis shows that an HP graph is HP-irreducible if it does not contain any tadpole subgraphs. By a graph with a tadpole we mean a graph with a subgraph which contains two lines coming from the same vertex so that after removing of these two lines the remaining subgraph becomes disconnected with the rest of the graph.

In [22] it has been shown that if one considers some special approximation for the planar theory, namely so called HP approximation, then Green's functions of the one-matrix model in this approximation coincide with correlation functions in a corresponding Boltzmann theory,

$$\Pi_{2n}^{appr}(g) = \langle \phi^{2n} (1 + g\phi^4)^{-1} \rangle. \quad (2.8)$$

In the LHS we select only irreducible HP graphs (this approximation we call the HP approximation) and in the RHS we omit all graphs with tadpoles. Let us sketch a pure combinatorial proof of equation (2.8) [25]. It is based on the following two statements. The first one states that the planar correlation functions without vacuum insertions in all order of perturbation theory are represented by a sum of all topologically non-equivalent graphs without any combinatorial factors. According the second one, all graphs in the Boltzmann theory contribute into correlations functions (2.3) without any combinatorial factors.

Now let us compare numerically the HP ap-

proximation with the planar approximation. The explicit formulas for an arbitrary planar Green's functions are well known [5]. On the Table we give the results of numerical calculations of the HP Green's functions  $G_2$ ,  $G_4$  and the planar Green's function  $\Pi_2$ ,  $\Pi_4$  for the same values of the coupling constant  $g$ . One can see that the answers for HP Green's functions  $G_2$ ,  $G_4$  practically saturate the planar Green's functions  $\Pi_2$ ,  $\Pi_4$  in the board interval of the values of  $g$ .

It is instructive to reformulate the Boltzmann theory so that it reproduces only tadpole-free HP graphs. For this purpose let us defined the fields  $\psi$  and  $\phi$  as follows  $\phi = a + a^+$ ,  $\psi = b + b^+$ , where  $a$ ,  $a^+$ ,  $b$  and  $b^+$  satisfy the following relations

$$aa^+ = 1, \quad bb^+ = 1, \quad ab^+ = 0, \quad ba^+ = 0.$$

This algebra has realization in the Boltzmannian Fock space under the vacuum  $|0\rangle$ :  $a|0\rangle = b|0\rangle = 0$ .

Now let us consider the following Green's functions

$$F_n = \langle 0 | \psi \phi^{n-2} \psi (1 + S_{int})^{-1} | 0 \rangle,$$

$$S_{int} = g\psi : \phi\phi : \psi = g\psi\phi\phi\psi - g\psi\psi.$$

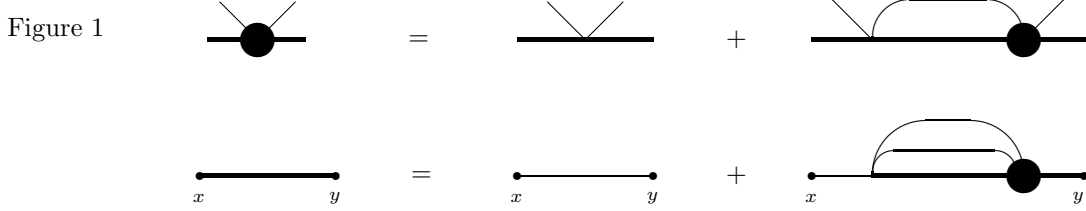
The Schwinger-Dyson equations for the Green's functions  $F_2$  and  $F_4$  have the form

$$F_2 = 1 - gF_4 + gF_2, \quad F_4 = -gF_2F_4 + F_2.$$

From these equations we find

$$F_2 = \frac{-1 + g + \sqrt{1 + 2g - 3g^2}}{2g(1 - g)},$$

$$F_4 = \frac{1 + g - \sqrt{1 + 2g - 3g^2}}{2g^2}.$$



### 3. Boltzmann Correlation Functions for $D$ -Dimensional Space-Time

In this section we derive the Schwinger-Dyson equations for Boltzmann correlation functions in  $D$ -dimensional Euclidean space. To avoid problems with tadpoles let us consider the two-field formulation. We adopt the following notations. Let  $\psi(x) = \psi^+(x) + \psi^-(x)$ ,  $\phi(x) = \phi^+(x) + \phi^-(x)$  be the Boltzmann fields with creation and annihilation operators satisfying the relations

$$\psi^-(x)\psi^+(y) = \phi^-(x)\phi^+(y) = D(x, y),$$

$$\psi^-(x)\phi^+(y) = \phi^-(x)\psi^+(y) = 0,$$

where  $D(x, y) = \int \frac{d^D k}{(2\pi)^D} (k^2 + m^2)^{-1} e^{ik(x-y)}$  is  $D$ -dimensional Euclidean propagator. The  $n$ -point Green's function is defined by

$$F_n(x_1, \dots, x_n) = \langle 0 | \psi(x_1) \phi(x_2) \dots \rangle \quad (3.9)$$

$$\phi(x_{n-1}) \psi(x_n) (1 + \int d^D x g \psi : \phi \phi : \psi)^{-1} | 0 \rangle.$$

Let us write down the Schwinger-Dyson equations for the two- and four-point correlation functions. We have

$$\begin{aligned} (-\Delta + m^2)_x F_2(x, y) &= g D(x, x) F_2(x, y) \\ &- g F_4(x, x, x, y) + \delta(x - y), \end{aligned} \quad (3.10)$$

$$\begin{aligned} (-\Delta + m^2)_y F_4(x, y, z, t) &= \\ &- g F_4(y, y, z, t) F_2(x, y) + \delta(y - z) F_2(x, t). \end{aligned} \quad (3.11)$$

Here we also assume that all vacuum insertions are dropped out. We see that equation (3.11) does not contain six-point correlation functions.

As a consequence we have a closed set of equations which are enough to find  $F_2$  and  $F_4$ . We define an one-particle irreducible (1PI) 4-point function  $\Gamma_4(x, y, z, t)$  as

$$\Gamma_4(x, y, z, t) = \int dx' dy' dz' dt' F_2^{-1}(x, x') \quad (3.12)$$

$$D^{-1}(y, y') D^{-1}(z, z') F_2^{-1}(t, t') \mathcal{F}_4(x', y', z', t'),$$

where  $\mathcal{F}_4$  is a connected part of  $F_4$

$$F_4(x, y, z, t) = \mathcal{F}_4(x, y, z, t) + F_2(x, t) D(y, z).$$

Note that in the contrast to the usual case in the RHS of (3.12) we multiply  $\mathcal{F}_4$  only on two full 2-point Green functions while in the usual case to get an 1PI Green function one multiplies an  $n$ -point Green function on  $n$  full 2-point functions. From (3.10) and (3.12) we have

$$\begin{aligned} \Gamma_4(p, k, r) &= -g - g \int dk' F_2(p + k - k') \times \\ &D(k') \Gamma_4(p + k - k', k', r) \end{aligned} \quad (3.13)$$

Equation (3.13) is the Bethe-Salpeter-like equation with the kernel which contains an unknown function  $F_2$ . As in the usual case we can write down  $F_2$  in term of the self-energy function  $\Sigma_2$

$$F_2 = \frac{1}{p^2 + m^2 + \Sigma_2}$$

and write equation (3.10) as an equation for  $\Sigma_2$ ,

$$\begin{aligned} \Sigma_2(p) &= g \int dk dq F_2(k) D(q) \times \\ &D(p - k - q) \Gamma_4(p, k, q). \end{aligned} \quad (3.14)$$

Equation (3.14) is similar to the usual relation between the self-energy function  $\Sigma_2$  and the 4-point vertex function for  $\varphi^4$  field theory, meanwhile equation (3.13) is specific for the Boltzmann field theory. Equations (3.13) and (3.14) are drawn on Fig.1.

Figure 2



Equations for  $\Gamma_4$  and  $\Sigma_2$  contain divergences. To remove them we apply to Boltzmannian Green's functions (3.9) the standard renormalization procedure based on  $R$ -operation. The perturbative expansion of the Boltzmannian Green's functions (3.9) is represented by the sum of HP Feynman graphs. We draw  $\phi$ -propagator by thin lines and  $\psi$ -propagator by thick lines. Only two- and four-point graphs may be divergent. Any two-point subgraph of the HP graph is also from the set of HP graphs and performing contractions of the two-point subgraphs of the HP graph one gets again an HP graph. But four-point subgraphs of an HP graph may be of HP type and may be not. We refer to the later case as to the case of  $\Pi$ -subgraphs. Examples of divergent HP-subgraphs and  $\Pi$ -subgraphs are drawn on Fig. 2a and 2b, respectively.

A detailed consideration [25] shows that all divergent parts can be collected to counterterms. Four-point divergent parts of  $\Pi$ -type require a counterterm :  $\psi\psi : \psi^-\psi^-$ .

Let us assume dimensional regularization and the minimal subtraction scheme with a scale  $\mu$ . For renormalized correlation functions we have

$$F_n(x_1, \dots, x_n; M(\mu), g(\mu), \lambda(\mu), \mu) =$$

$$\langle 0 | \psi(x_1) \phi(x_2) \dots \phi(x_{n-1}) \psi(x_n) \times$$

$$(1 + \int d^D x [\mathcal{L}_{int}(\psi, \phi) + \mathcal{L}_{ct}(\psi, \phi)])^{-1} | 0 \rangle,$$

where

$$\mathcal{L}_{int} = (M^2 - m^2)\psi^2 +$$

$$\mu^\varepsilon g \psi : \phi\phi : \psi + \mu^\varepsilon \lambda : \psi\psi : \psi^-\psi^-,$$

$$\mathcal{L}_{ct} = (Z_\psi - 1)(\partial_\mu \psi)^2 + \delta M^2 \psi^2 +$$

$$\mu^\varepsilon \delta g \psi : \phi\phi : \psi + \mu^\varepsilon \delta \lambda : \psi\psi : \psi^-\psi^-$$

and  $\varepsilon = 4 - D$ . Here we add a new term to the interaction. This  $\lambda$ -term modifies only the mass term in equation (3.10) (see [25] for more details).

The renormalized correlations functions satisfy the following renormalization group equation

$$(\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \beta_\lambda \frac{\partial}{\partial \lambda} - \gamma_M \frac{\partial}{\partial \ln M^2} + \gamma) \times$$

$$F_n(x_1, \dots, x_n; M(\mu), g(\mu), \lambda(\mu), \mu) = 0, \quad (3.15)$$

where

$$\beta_g = \mu \frac{\partial g(\mu)}{\partial \mu}, \quad \beta_\lambda = \mu \frac{\partial \lambda(\mu)}{\partial \mu},$$

$$\gamma_M = -\mu M^{-2} \frac{\partial M^2(\mu)}{\partial \mu}, \quad \gamma = \mu \frac{\partial \ln Z_\psi(\mu)}{\partial \mu}.$$

The difference between (3.15) and the usual renormalization group equation is that the anomalous dimension  $\gamma$  in (3.15) is not multiplied on  $n/2$ . There are also a differences in the expressions for the beta-functions in terms of the counterterms  $\delta g$ ,  $\delta \lambda$  and  $Z_\psi - 1$ . In dimensional regularization the counterterms are poles in  $\varepsilon$ ,

$$\delta g = \sum_{i=1}^{\infty} \frac{a_i(g, \lambda)}{\varepsilon^i}, \quad \delta \lambda = \sum_{i=1}^{\infty} \frac{b_i(g, \lambda)}{\varepsilon^i},$$

$$Z_\psi = 1 + \sum_{i=1}^{\infty} \frac{c_i(g, \lambda)}{\varepsilon^i}. \quad (3.16)$$

As in the usual case one can write the low-order expression for the  $\beta$ -function in terms of  $c_1$  and

$a_1$ . But now there is a new expression for  $g_0$  in terms of  $\delta g$  and  $Z_\psi$ . We have

$$g_0 = \mu^\varepsilon (g + \delta g) Z_\psi^{-1},$$

$$\lambda_0 = \mu^\varepsilon (\lambda + \delta \lambda) Z_\psi^{-2}. \quad (3.17)$$

Applying  $\mu \frac{d}{d\mu}$  to both sides of (3.17) one finds

$$\beta_g(g) = (1 - \frac{\partial}{\partial \ln g})(g c_1(g, \lambda) - a_1(g, \lambda)).$$

$$\beta_\lambda(\lambda) = (1 - \frac{\partial}{\partial \ln \lambda})(2\lambda c_1(g, \lambda) - b_1(g, \lambda)).$$

Taking into account the explicit form of  $c_1(g, \lambda)$ ,  $a_1(g, \lambda)$  and  $b_1(g, \lambda)$  we get

$$\beta_g(g, \lambda) = \frac{g^2}{8\pi^2} + \frac{g^3}{(16\pi^2)^2} + O(g^4, \lambda^4),$$

$$\beta_\lambda(g, \lambda) = \frac{\lambda^2}{8\pi^2} + O(g^4, \lambda^4). \quad (3.18)$$

Note that there is only a numerical difference of the HP  $\beta$  function with the standard  $\beta$  function.

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